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A Fisher/KPP-type equation with density-dependent diffusion and convection: travelling-wave solutions

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Abstract
This paper concerns processes described by a nonlinear partial differential equation that is an extension of the Fisher and KPP equations including density-dependent diffusion and nonlinear convection. The set of wave speeds for which the equation admits a wavefront connecting its stable and unstable equilibrium states is characterized. There is a minimal wave speed. For this wave speed there is a unique wavefront which can be found explicitly. It displays a sharp propagation front. For all greater wave speeds there is a unique wavefront which does not possess this property. For such waves, the asymptotic behaviour as the equilibrium states are approached is determined.

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1. Introduction

A characteristic of a huge number of biological, chemical and physical phenomena is that in the course of time a spatio-temporal pattern develops from a state that does not initially exhibit any structure. In many instances, the population density or concentration will evolve into a spatial profile which does not appear to change shape with time, yet moves with a well-defined velocity. By its very nature, such a phenomenon indicates the formation of a travelling wave. One of the many challenges involved in mathematically modelling biological, chemical, physical and other processes is identifying whether or not the model can simulate the occurrence of such a wave. Should this be the case, there is the further challenge of predicting the shape and velocity of the travelling wave, and relating this to the spatio-temporal pattern being observed in practice.

Many models in the form of nonlinear partial differential equations—the linear diffusion equation with logistic growth is an example *par excellence*—admit travelling-wave solutions with a continuous spectrum of wave speeds. In such situations, the task of identifying the travelling wave being observed is far from simple. One has in some way to determine the range of admissible speeds, and to find the features that distinguish one travelling wave from another, as precisely as possible.

The aim of this paper is to carry out the just-described programme for the substantial model embodied in the partial differential equation

$$u_t + bu^k u_x = (au^k u_x)_x + cu(1 - u^k), \tag{1}$$

where $a \geqslant 0, b, c \geqslant 0$, and $k > 0$ are constants. The unknown (density or concentration) u is non-negative, x is a space coordinate, and t denotes time. Subscripts denote partial differentiation with respect to the relevant variable. The first term on the right-hand side of the equation corresponds to a diffusive process with a diffusion coefficient which depends on the unknown. The second term on the left-hand side represents convection with a velocity function which likewise depends on the unknown. The last term in the equation is a reaction term.

Equation (1) arises in the study of pattern formation by bacterial colonies exemplified by the growth of bacteria of the type *Paenbacillus dendritiformis* on a thin layer of agar in a Petri dish [1]. These bacteria cannot move on a dry surface, and produce a layer of lubricating fluid in which they swim. In a uniform layer of liquid, bacterial swimming is a random process which can be approximated by diffusion. The lubrication fluid flows by convection caused by motion of the bacteria and diffusion. The availability of nutrients affects the reproduction of bacteria, the production of lubricating fluid, and the withdrawal of bacteria into a pre-pore state. The bacteria consume the nutrients. A continuum approach to the dynamics leads to a model comprising three coupled reaction–diffusion equations with unknowns: the density of the bacteria u, the height of the lubrication layer v, and the available nutrient w, respectively. Considering the density of bacteria in the pre-pore state optionally gives rise to a fourth equation. Under simplifying assumptions, the variable v can be eliminated. This leads [1, p 242] to the equation for the bacterial field $u_t = \nabla \cdot (au^k \nabla u) + \mathsf{c}$, where ∇ denotes the standard differential operator in two space dimensions, and c the nett effect of reproduction and withdrawal of bacteria. The term c vanishes when $u = 0$, and depends monotonically on w in such a way that it is a source when w exceeds some critical nutrient level and a sink when w falls below this level. When chemotaxis is taken into account, the equation becomes

$$u_t = \nabla \cdot (au^k \nabla u - b_0 u^{k+1} X(z) \nabla z) + \mathsf{c},$$

where z denotes the concentration of the chemical responsible for the chemotaxis, $X(z)\nabla z$ the chemical gradient sensed by the bacteria, and b_0 a constant which is positive for attraction and negative for repulsion [1, p 243]. Assuming that the motion is unidimensional, the chemical gradient is uniform, and $\mathsf{c} := cu(1-u^k)$ gives rise to equation (1). This reaction term embodies the properties described previously under the simplifying assumption that the nutrient level is appropriately related to the bacterial density. The value of the bacterial density corresponding to the critical nutrient level is normalized to $u = 1$.

With one or more terms omitted, and with one or more terms replaced by alternative expressions, equation (1) is found in many other areas of application including chemical reaction, combustion, thermal waves in plasma, population dynamics, and ecology [2–5].

Mathematically, equation (1) may be classified as of quasilinear degenerate second-order parabolic type for $a > 0$, and, of quasilinear first-order type for $a = 0$ and $b \neq 0$. Moreover, in both instances, when $c = 0$ it is a nonlinear conservation law. As special cases, the equation includes the Fisher/KPP equation with nonlinear diffusion ($a > 0, b = 0, c > 0$), the degenerate Burgers equation ($a > 0, b \neq 0, c = 0$), and, a first-order partial differential equation with reaction ($a = 0, c > 0$). When $a > 0$ and $b \neq 0$, one may normalize the equation by suitably scaling the independent variables x and t so that $a = b = 1$. Similarly, if $a > 0$ and $c > 0$ one may scale the independent variables so that $a = c = 1$, and, if $b \neq 0$ and $c > 0$ one may normalize the equation in such a way that $b = c = 1$.

Equation (1) admits two equilibrium solutions when $c > 0$, namely an unstable solution $u \equiv 0$ and a stable one $u \equiv 1$. In order to study the propagation of the unstable state into the stable one, we look for travelling-wave (TW) solutions

$$u(x,t) = f(\xi), \quad \text{where} \quad \xi = x - \sigma t \tag{2}$$

and σ is the *wave speed*, satisfying

$$f(\xi) \to 0 \quad \text{as} \quad \xi \to \infty \tag{3}$$

and

$$f(\xi) \to 1 \quad \text{as} \quad \xi \to -\infty. \tag{4}$$

Substituting (2) into (1) leads to the ordinary differential equation

$$(af^k f')' + (\sigma - bf^k)f' + cf(1 - f^k) = 0. \tag{5}$$

Practically all the main questions related to such TWs were formulated already in 1937 [6, 7] in connection with Fisher/KPP equation,

$$u_t = u_{xx} + u(1-u). \tag{6}$$

(1) Are there σ for which problem (3)–(5) has a solution?
(2) If the answer to the previous question is yes for all $\sigma \geqslant \sigma_c$, what is the value of the minimal speed σ_c?
(3) Is the TW with minimal speed an attractor?

The answer for equation (6) is the following: a TW exists if and only if $\sigma \geqslant 2$, i.e. $\sigma_c = 2$, and, for appropriate classes of initial data the solution of the initial-value problem converges to the TW with minimal speed as $t \to \infty$ in both form and speed.

The answer to the first two questions above has been extended to the more general equation

$$u_t + buu_x = u_{xx} + u(1-u). \tag{7}$$

There holds $\sigma_c = 2$ for $b \leqslant 2$ and $\sigma_c = b/2 + 2/b$ for $b > 2$ [2, 5]. To the best of the authors' knowledge, the answer to the third question for $b \neq 0$ is not yet known.

The equation

$$u_t = (u^k u_x)_x + u(1-u^k) \tag{8}$$

is rather well studied at present. The first two questions have been treated in [5, 8–13], the third one for different classes of initial data in [13–15]. Further results on equation (8) can be found in [16]; for instance, it can be transformed to a purely diffusive process in an inhomogeneous medium.

In this paper we address the impact of nonlinear convection on the nonlinear diffusion–reaction process (8), dealing with the first two questions concerning TWs. The third question is investigated in [15].

Our main result is the following.

Theorem 1. *Suppose that $a > 0$, $c > 0$ and $k > 0$. Let*

$$\sigma^* := \frac{b + \sqrt{b^2 + 4(k+1)ac}}{2(k+1)}.$$

Then, modulo translation, problem (3)–(5) has a unique solution f for every $\sigma \geqslant \sigma^*$ and no solution for $\sigma < \sigma^*$.

(a) *When* $\sigma = \sigma^*$, f *is a sharp front, i.e. the support of* f *is bounded above, and, modulo translation,*

$$f(\xi) = \begin{cases} \left[1 - \exp\left(\frac{\sigma k}{a}\xi\right)\right]^{1/k} & \textit{for} \quad \xi < 0 \\ 0 & \textit{for} \quad \xi \geqslant 0. \end{cases}$$

(b) *When* $\sigma > \sigma^*$, f *is positive, strictly monotonic,*

$$(\ln\{1 - f\})'(\xi) \to \frac{2kc}{\sigma - b + \sqrt{(\sigma - b)^2 + 4kac}} \quad \textit{as} \quad \xi \to -\infty,$$

and,

$$(\ln f)'(\xi) \to -\frac{c}{\sigma} \quad \textit{as} \quad \xi \to \infty.$$

Corollary 1. *Suppose that* $a > 0$, $c > 0$ *and* $k > 0$. *Let*

$$\sigma^{**} := \frac{b - \sqrt{b^2 + 4(k+1)ac}}{2(k+1)}.$$

Then, modulo translation, equation (5) has a unique solution f *satisfying*

$$f(\xi) \to 0 \quad \textit{as} \quad \xi \to -\infty \qquad \textit{and} \qquad f(\xi) \to 1 \quad \textit{as} \quad \xi \to \infty \tag{9}$$

for every $\sigma \leqslant \sigma^{**}$ *and no such solution for* $\sigma > \sigma^{**}$.

(a) *When* $\sigma = \sigma^{**}$, f *is a sharp front, i.e. the support of* f *is bounded below, and, modulo translation,*

$$f(\xi) = \begin{cases} \left[1 - \exp\left(-\frac{|\sigma| k}{a}\xi\right)\right]^{1/k} & \textit{for} \quad \xi > 0 \\ 0 & \textit{for} \quad \xi \leqslant 0. \end{cases}$$

(b) *When* $\sigma < \sigma^{**}$, f *is positive, strictly monotonic,*

$$(\ln f)'(\xi) \to \frac{c}{|\sigma|} \quad \textit{as} \quad \xi \to -\infty,$$

and,

$$(\ln\{1 - f\})'(\xi) \to -\frac{2kc}{|\sigma| + b + \sqrt{(|\sigma| + b)^2 + 4kac}} \quad \textit{as} \quad \xi \to \infty.$$

Remark 1. The TWs with $\sigma > \sigma^*$ are classical solutions (even real analytic). The sharp front with minimal speed $\sigma = \sigma^*$ is not smooth. In particular, when $k \geqslant 1$, it is merely Hölder continuous with exponent $1/k$ at $\xi = 0$. Nonetheless, f^{k+1} is continuously differentiable everywhere. So, in terms of applications, the flux is continuous, and this TW is a weak solution in the usual mathematical sense.

Remark 2. The result holds for $b = 0$. In this case, $\sigma^* = \sqrt{ac/(k+1)}$.

Remark 3 (conjecture)**.** Considering the Cauchy problem for equation (1) with initial data that are non-negative, not identically zero, and have compact support, theorem 1 and its corollary suggest that the solution will be such that its positivity set with respect to the spatial variable becomes connected, with a lower boundary which moves to the left with a speed which approaches σ^{**} and an upper boundary which moves to the right with a speed which approaches σ^*, as time tends to infinity. Comparison principle arguments using the TWs readily confirm that the positivity set cannot grow any faster.

Table 1. Parameters which determine the asymptotic behaviour of a TW of equation (10) whereby σ_c and b^* are as stated in proposition 1.

σ	$m+p$	b	β	Φ
$\sigma=\sigma_c=0$	$m+p>2k+1$	$b<b^*$	$m+p-k-1$	$\frac{c}{\lvert b\rvert}$
		$b=b^*$	k	$\frac{\lvert b\rvert}{(k+1)a}$
	$m+p=2k+1$	$b\leqslant b^*$	k	$\frac{\lvert b\rvert-\sqrt{b^2-(b^*)^2}}{2(k+1)a}$
$\sigma=\sigma_c>0$	$m+p>1$		0	$\frac{\sigma}{a}$
	$m+p=1$		0	$\frac{\sigma+\sqrt{\sigma^2-4ac}}{2a}$
$\sigma>\sigma_c$	$m+p>1$		$m+p-1$	$\frac{c}{\sigma}$
	$m+p=1$		0	$\frac{\sigma-\sqrt{\sigma^2-4ac}}{2a}$

Results in analogy to theorem 1 can be obtained for the more general equation

$$u_t + bu^k u_x = (au^m u_x)_x + cu^p(1-u^q) \tag{10}$$

with $a>0, b, c>0, k>0, m, p$, and $q>0$ constants.

Proposition 1. *Suppose that $a>0, c>0, k>0$ and $q>0$. When $m+p<1$, equation (10) has no TWs satisfying (3) and (4). When $m+p\geqslant 1$, there is such a TW if and only if $\sigma\geqslant\sigma_c$ for some number σ_c which depends only on $ac, b, k, m+p$ and q; in which case, modulo translation, the TW f is unique. With regard to the minimal speed, if $m+p\geqslant 2k+1$, there exists a number $b^*<0$, which depends only on $ac, k, m+p$ and q, such that $\sigma_c=0$ for $b\leqslant b^*$ and $\sigma_c>0$ for $b>b^*$. On the other hand, if $m+p<2k+1$, necessarily $\sigma_c>0$. In particular, if $m+p=1$, then $\sigma_c\geqslant 2\sqrt{ac}$. Concerning the asymptotic behaviour of the TW,*

$$(\ln\{1-f\})'(\xi) \to \frac{2qc}{\sigma-b+\sqrt{(\sigma-b)^2+4qac}} \quad as \quad \xi\to-\infty,$$

and, with $\beta\geqslant 0$ and $\Phi>0$ as shown in table 1, the following holds.

(a) If $m>\beta$, f is a sharp front, and,

$$(f^{m-\beta})'(\xi)\to-(m-\beta)\Phi \quad as \quad \xi\uparrow\xi^*,$$

where ξ^* denotes the least upper bound of the support.

(b) If $m=\beta$, f is positive, strictly monotonic and,

$$(\ln f)'(\xi)\to-\Phi \quad as \quad \xi\uparrow\infty.$$

(c) If $m<\beta$, f is positive, strictly monotonic, and,

$$(f^{-(\beta-m)})'(\xi)\to(\beta-m)\Phi \quad as \quad \xi\uparrow\infty.$$

The outstanding feature of (1) is that for this specific instance of equation (10), both the minimal speed σ_c and the corresponding TW can be obtained explicitly.

Other instances of equation (10) for which the minimal speed and a TW can be found explicitly are given below. The rider 'for $\xi<0$' implies that the TW f is a sharp front, with $f(\xi)=0$ for $\xi\geqslant 0$.

Proposition 2. *For any equation of the form*

$$u_t + bu^k u_x = (au^m u_x)_x + cu^{k+1-m}(1-u^k),$$

there holds $\sigma_c=\sigma^*$.

(i) When $m = k/2$ *and* $\sigma = \sigma^*$,

$$f(\xi) = \left[\tanh\left(-\frac{\sigma k}{2a}\xi\right)\right]^{2/k} \quad \textit{for} \quad \xi < 0.$$

(ii) When $m = 0$ *and* $\sigma = \sigma^*$,

$$f(\xi) = \left[1 + \exp\left(\frac{\sigma k}{a}\xi\right)\right]^{-1/k}.$$

Proposition 3. *For*

$$u_t + bu^k u_x = (au^m u_x)_x + cu^{1-m}(1 - u^k),$$

there holds

$$\sigma_c = \begin{cases} 2\sqrt{ac} & \textit{if} \quad b \leqslant (k+1)\sqrt{ac} \\ b/(k+1) + (k+1)ac/b & \textit{if} \quad b > (k+1)\sqrt{ac}. \end{cases}$$

(i) When $m = k$, $b > 0$ *and* $\sigma = b/(k+1) + (k+1)ac/b$,

$$f(\xi) = \left[1 - \exp\left(\frac{kb}{(k+1)a}\xi\right)\right]^{1/k} \quad \textit{for} \quad \xi < 0.$$

(ii) When $m = k/2$, $b > 0$ *and* $\sigma = b/(k+1) + (k+1)ac/b$,

$$f(\xi) = \left[\tanh\left(-\frac{kb}{2(k+1)a}\xi\right)\right]^{2/k} \quad \textit{for} \quad \xi < 0.$$

(iii) When $m = 0$, $b > 0$ *and* $\sigma = b/(k+1) + (k+1)ac/b$,

$$f(\xi) = \left[1 + \exp\left(\frac{kb}{a(k+1)}\xi\right)\right]^{-1/k}.$$

The above proposition extends the results mentioned for equation (7).

Proposition 4. *For*

$$u_t + bu^k u_x = (au^m u_x)_x + cu^{1-m}(1 - u^{2k}),$$

there holds

$$\sigma_c = \begin{cases} 2\sqrt{ac} & \textit{if} \quad b \leqslant k\sqrt{ac} \\ \sigma^* + ac/\sigma^* & \textit{if} \quad b > k\sqrt{ac}. \end{cases}$$

(i) When $m = k$ *and* $\sigma = \sigma^* + ac/\sigma^*$,

$$f(\xi) = \left[1 - \exp\left(\frac{\sigma^* k}{a}\xi\right)\right]^{1/k} \quad \textit{for} \quad \xi < 0.$$

(ii) When $m = k/2$ *and* $\sigma = \sigma^* + ac/\sigma^*$,

$$f(\xi) = \left[\tanh\left(-\frac{\sigma^* k}{2a}\xi\right)\right]^{2/k} \quad \textit{for} \quad \xi < 0.$$

(iii) When $m = 0$ *and* $\sigma = \sigma^* + ac/\sigma^*$,

$$f(\xi) = \left[1 + \exp\left(\frac{\sigma^* k}{a}\xi\right)\right]^{-1/k}.$$

2. The integral equation method

Until recently the only line of attack on problems such as (3)–(5) was phase-plane analysis. The first convincing illustration is the previously mentioned paper [7] of 1937. The substitution $P(f) = -af^k f'$ in (5) leads to

$$P' = \sigma - bf^k - ac\frac{f^{k+1}(1-f^k)}{P},$$

and one has to deal with a very strong possible non-existence and non-uniqueness (the right-hand side is not only not Lipschitz continuous with respect to P, it even becomes unbounded as $P \to 0$). We transfer this singular equation to an integral equation and deal with it in a purely analytic way. As a first step, we show the following.

Lemma 1. *Any solution f of problem (3)–(5) is monotonic and such that*

$$a(f^{k+1})'(\xi) \to 0 \quad \textit{as} \quad \xi \to \pm\infty. \tag{11}$$

To justify the above, we adapt a standard argument [17]. Recalling that we are dealing with the case $a > 0$ and $c > 0$, we write (5) in the form

$$F' + cf(1-f^k) = 0, \tag{12}$$

where

$$F := \frac{a}{k+1}(f^{k+1})' + \sigma f - \frac{b}{k+1}f^{k+1}, \tag{13}$$

and assume that f^{k+1} and F are continuously differentiable. We note that since f is non-negative and satisfies (3), necessarily $0 \leqslant f(\xi) \leqslant 1$ for all sufficiently large ξ. So, by (12), F is non-increasing for all such ξ. Combining this deduction with (3), it follows from (13) that $(f^{k+1})'(\xi) \to L$ as $\xi \to \infty$ for some $-\infty \leqslant L < \infty$. However, integrating this deduction, it is easily checked that it violates (3) unless $L = 0$. Thus, the part of (11) relating to $\xi \to \infty$ is justified. Moreover, $F(\xi) \to 0$ as $\xi \to \infty$. Next, we observe that the derivatives f' and f'' exist classically wherever $f > 0$. Hence, if $f' = 0$ at any point where $f > 0$, (5) implies that $af^k f'' = -cf(1-f^k)$. So, f cannot have a maximum where $f > 1$, nor a minimum where $0 < f < 1$. Furthermore, f cannot have an extremum of either kind where $f = 1$, since given $f' = 0$ and $f = 1$, local uniqueness for solutions of ordinary differential equations such as (5) implies $f \equiv 1$, and this violates (3). Taking the previous deductions together, f will be monotonic unless there exists an interval (ξ_0, ξ_1) with $-\infty < \xi_0 < \xi_1 \leqslant \infty$, such that $0 < f < 1$ on (ξ_0, ξ_1), $f(\xi_0) = F(\xi_0) = 0$, and, $f(\xi) \to 0$ and $F(\xi) \to 0$ as $\xi \uparrow \xi_1$. However, in this event, integrating (12) from ξ_0 to ξ_1, yields $\int_{\xi_0}^{\xi_1} cf(1-f^k)\,\mathrm{d}\xi = 0$. This gives a contradiction, wherewith the monotonicity is proven. Hereafter, (12) implies that F is non-increasing everywhere. From this and (4) it follows that $(f^{k+1})'(\xi) \to L$ as $\xi \to -\infty$ for some $-\infty < L \leqslant \infty$. Arguing as before necessitates $L = 0$. Herewith, the lemma is justified.

Since any solution of problem (3)–(5) is monotonic on $(-\infty, \infty)$, it is a *wavefront*. Therewith it falls within the scope of a theory developed in [5, 18, 19] for studying monotonic TWs of reaction–convection–diffusion equations of the class

$$u_t = (\mathsf{a}(u))_{xx} + (\mathsf{b}(u))_x + \mathsf{c}(u) \tag{14}$$

using a singular integral equation. The hypotheses required of the coefficients in this theory are minimal. Notwithstanding, rather than describing the full theory here, we confine ourselves to those results which are relevant to the problem in hand.

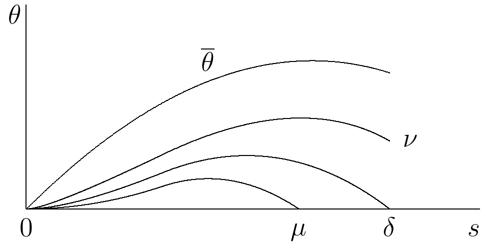

Figure 1. One parameter family of solutions of the integral equation.

Suppose that a is continuously differentiable on [0, 1] with $\mathsf{a}'(u) > 0$ for $0 < u < 1$, that b is continuously differentiable on [0, 1] with $\mathsf{b}(0) = 0$, and, c is continuous on [0, 1] with $\mathsf{c}(0) = \mathsf{c}(1) = 0$ and $\mathsf{c}(u) > 0$ for $0 < u < 1$. Under these hypotheses, it has been shown [5, 18, 19] that the set of wave speeds σ for which equation (14) admits a wavefront solution f satisfying (3) and (4) is either empty or an interval $[\sigma_c, \infty)$ for some number σ_c. Furthermore, in the latter case, for every $\sigma \in [\sigma_c, \infty)$ this wavefront solution is unique except with respect to translation. The foundation of these results is the deeper finding that equation (14) admits a wavefront solution satisfying (3) and (4) if and only if the integral equation

$$\theta(s) = \sigma s + \mathsf{b}(s) - \int_0^s \frac{\mathsf{c}(r)\mathsf{a}'(r)}{\theta(r)}\,\mathrm{d}r \tag{15}$$

has a solution θ on [0, 1] which is positive on (0, 1) and such that $\theta(1) = 0$. Assuming that $f(0) = \alpha$ for some $0 < \alpha < 1$, the wavefront solution f is then given by

$$\begin{aligned} &f(\xi) = 1 \quad \text{for} \quad \xi \leqslant \xi^-, \\ &\int_{f(\xi)}^{\alpha} \frac{\mathsf{a}'(s)}{\theta(s)}\,\mathrm{d}s = \xi \quad \text{for} \quad \xi^- < \xi < \xi^+, \end{aligned} \tag{16}$$

and

$$f(\xi) = 0 \quad \text{for} \quad \xi \geqslant \xi^+, \tag{17}$$

where

$$\xi^- = -\int_\alpha^1 \frac{\mathsf{a}'(s)}{\theta(s)}\,\mathrm{d}s \qquad \text{and} \qquad \xi^+ = \int_0^\alpha \frac{\mathsf{a}'(s)}{\theta(s)}\,\mathrm{d}s.$$

This finding can be interpreted in terms of the traditional approach discussed at the start of this section, by the identification of $P(f)$ with $\theta(s)$. In this context, (11) is equivalent to $\theta(0) = \theta(1) = 0$. The wavefront f will be a sharp front if and only if ξ^+ is finite.

With regard to solving (15), under the hypotheses stated, the following is known. Either the equation has no solution, or it has a one parameter family of solutions which contains a member that can characterized as maximal. To be more precise, let us suppose that the equation has at least one solution on an interval $[0, \delta]$ for some $0 < \delta \leqslant 1$. Then, in this event, it has a particular solution $\overline{\theta}$ on $[0, \delta]$ with the following properties. Given any $0 < \mu < \delta$, (15) has a unique solution on $[0, \mu]$ that is positive on $(0, \mu)$ and vanishes at $s = \mu$. Given any $0 \leqslant \nu \leqslant \overline{\theta}(\delta)$, (15) has a unique solution on $[0, \delta]$ that is positive on $(0, \delta)$ and equal to ν at $s = \delta$. Equation (15) has no other solutions. Except in 0, none of the aforementioned solutions intersect. Consequently, they are successively ordered by the parameters μ and ν, and, with respect to this ordering, $\overline{\theta}$ is maximal. See figure 1.

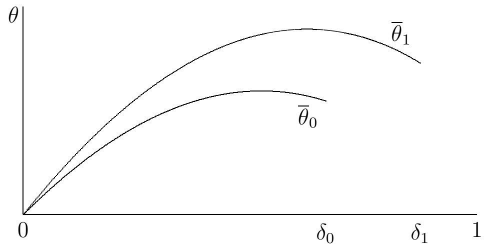

Figure 2. Ordering of maximal solutions of the integral equation.

On top of the above, if for some number σ_0, equation (15) has a solution on an interval $[0, \delta_0]$ with $0 < \delta_0 \leqslant 1$, then for any number $\sigma_1 > \sigma_0$ the equation has a solution on an interval $[0, \delta_1]$ with $\delta_0 \leqslant \delta_1 \leqslant 1$, and, $\delta_1 > \delta_0$ if $\delta_0 < 1$. Moreover, if $\overline{\theta}_0$ and $\overline{\theta}_1$ denote the corresponding maximal solutions there holds $\overline{\theta}_1 > \overline{\theta}_0$ on $(0, \delta_0]$. See figure 2.

It follows from the previous remarks that *the critical wave speed σ_c can be characterized as the smallest number σ for which* (15) *has a solution on* [0, 1]. Moreover, for any $\sigma > \sigma_c$, the solution θ that gives rise to the wavefront solution f cannot be the maximal solution of the integral equation.

3. The proof of the main theorem and its corollary

In the present context, the integral equation (15) reads

$$\theta(s) = \sigma s - \frac{b}{k+1} s^{k+1} - ac \int_0^s \frac{r^{k+1}(1-r^k)}{\theta(r)}\, \mathrm{d}r, \tag{18}$$

while (16) becomes

$$\int_{f(\xi)}^{\alpha} \frac{s^k}{\theta(s)}\, \mathrm{d}s = \frac{\xi}{a} \quad \text{for} \quad \xi^- < \xi < \xi^+, \tag{19}$$

with

$$\xi^- = -a \int_\alpha^1 \frac{s^k}{\theta(s)}\, \mathrm{d}s \quad \text{and} \quad \xi^+ = a \int_0^\alpha \frac{s^k}{\theta(s)}\, \mathrm{d}s. \tag{20}$$

The key to theorem 1 is an explicit solution of the particular integral equation (18) with fortuitous properties. This is the function

$$\theta^*(s) := \sigma^* s(1 - s^k).$$

By substitution it can easily be verified to solve (18) for $\sigma = \sigma^*$. From this and the general results described in the previous section, it follows immediately that the set of numbers σ for which (3)–(5) has a solution is an interval $[\sigma_c, \infty)$ with $\sigma_c \leqslant \sigma^*$, and, that for every $\sigma \geqslant \sigma_c$ problem (3)–(5) has a unique solution modulo translation. Furthermore, recalling (17), (19) and (20), when $\sigma = \sigma^*$ the solution is as stated in part (a) of the theorem.

To prove theorem 1 the tasks remaining are therefore to show that $\sigma_c = \sigma^*$ and that for $\sigma > \sigma^*$ the solution f has the behaviour indicated. To fulfil these tasks, we determine some additional properties of solutions of the integral equation (18).

The first property we use is that for any $\sigma > 0$, equation (18) has a maximal solution $\overline{\theta}$ on $[0, \delta]$ for some $0 < \delta \leqslant 1$, and,

$$\overline{\theta}(s) \sim \sigma s \quad \text{as} \quad s \downarrow 0. \tag{21}$$

To show this we fix $0 < \varepsilon < \sigma$ and consider the function $\vartheta(s) := (\sigma - \varepsilon)s$. It can be verified that ϑ satisfies (18) if $\sigma s - \frac{b}{k+1}s^{k+1}$ is replaced by

$$B(s) := (\sigma - \varepsilon)s + \frac{ac}{\sigma - \varepsilon}\left(\frac{s^{k+1}}{k+1} - \frac{s^{2k+1}}{2k+1}\right).$$

Moreover, $\sigma s - \frac{b}{k+1}s^{k+1} \geqslant B(s)$ for all $0 \leqslant s \leqslant \delta$, if $0 < \delta \leqslant 1$ is chosen small enough. So by a comparison principle for solutions of integral equations of type (15), our equation (18) has a maximal solution $\overline{\theta} \geqslant \vartheta$ on $[0, \delta]$. This gives the existence result, and $\liminf_{s\downarrow 0} \overline{\theta}(s)/s \geqslant \sigma$ in view of the arbitrariness of ε. On the other hand, it is immediate from (18) that any solution θ is such that $\limsup_{s\downarrow 0} \theta(s)/s \leqslant \sigma$. Together this yields (21).

The second property we employ is that given any $\delta > 0$ and $\gamma > 0$, equation (18) has at most one solution θ satisfying

$$\theta(s) \geqslant \gamma s \quad \text{for all} \quad 0 \leqslant s \leqslant \delta. \tag{22}$$

To see this, we suppose that there exists a $\gamma > 0$ and two solutions θ_1 and θ_2 on some interval $[0, \delta]$ such that $\theta_i(s) \geqslant \gamma s$ for $0 \leqslant s \leqslant \delta$ and $i = 1, 2$. Then using (18) we compute that

$$\begin{aligned}|\theta_1 - \theta_2|(s) &\leqslant ac\int_0^s \frac{r^{k+1}(1-r^k)\,|\theta_1(r) - \theta_2(r)|}{\theta_1(r)\theta_2(r)}\,\mathrm{d}r \\ &\leqslant ac\left(\frac{\delta^k}{k} - \frac{\delta^{2k}}{2k}\right)\frac{1}{\gamma^2}\max\{|\theta_1 - \theta_2|(r) : 0 \leqslant r \leqslant \delta\}\end{aligned}$$

for all $0 < s \leqslant \delta$. Hence, if δ is sufficiently small, $\theta_1 \equiv \theta_2$ on $[0, \delta]$. Thus, there cannot be two different solutions satisfying (18). It follows that for $\sigma > 0$ the maximal solution $\overline{\theta}$ is the only solution of (18) that satisfies an inequality of the type (22).

The last property of solutions of (18) that we need is the following. For $\sigma > 0$, any solution $\theta \not\equiv \overline{\theta}$ is such that

$$\theta(s) \sim \frac{ac}{\sigma}s^{k+1} \quad \text{as} \quad s \downarrow 0. \tag{23}$$

To confirm this, we consider the functions $\vartheta^{\pm}(s) := \frac{ac}{\sigma\mp\varepsilon}s^{k+1}$, where $0 < \varepsilon < \sigma$. By substitution it can be verified that each of these functions satisfies (18) if $\sigma s - \frac{b}{k+1}s^{k+1}$ is replaced by

$$B(s) := (\sigma \mp \varepsilon)s + \left(\frac{ac}{\sigma \mp \varepsilon} - \frac{\sigma \mp \varepsilon}{k+1}\right)s^{k+1}.$$

Furthermore, $s \mapsto \sigma s - \frac{b}{k+1}s^{k+1} - B(s)$ is strictly monotonic on $[0, \delta]$ if $0 < \delta \leqslant 1$ is small enough. Subsequently, by comparison principle arguments for the general integral equation (15), θ and $\vartheta^{\pm}$ can have at most one point of intersection on $(0, \delta]$ for such δ. Let us now suppose that $\theta > \vartheta^+$ on $(0, \delta_0)$ for some $0 < \delta_0 \leqslant \delta$. Then substituting $\theta(r) > \vartheta^+(r)$ in the right-hand side of (18), we deduce that $\theta(s) > \varepsilon s - \frac{b-\sigma+\varepsilon}{k+1}s^{k+1}$ for all $0 < s < \delta_0$. However, from the previous paragraph, we know that the maximal solution $\overline{\theta}$ is the only solution that can satisfy this inequality. Thus, we must have $\theta < \vartheta^+$ on $(0, \delta_1)$ for some $0 < \delta_1 \leqslant \delta$. Alternatively, supposing that $\theta < \vartheta^-$ on $(0, \delta_0)$ for some $0 < \delta_0 \leqslant \delta$, substituting $\theta(r) < \vartheta^-(r)$ on the right-hand side of (18), we deduce that $\theta(s) < -\varepsilon s - \frac{b-\sigma-\varepsilon}{k+1}s^{k+1}$ for all $0 < s < \delta_0$. This contradicts the positivity of $\theta(s)$ for small s. So, together, there holds $\vartheta^- < \theta < \vartheta^+$ on $(0, \delta_2)$ for some $0 < \delta_2 \leqslant \delta$. In view of the arbitrariness of ε, this gives (23).

We are now in a position to show that $\sigma_c = \sigma^*$. The argument is as follows. Suppose, to the contrary, that $\sigma_c < \sigma^*$. Then according to the general theory, the explicit solution $\theta^*(s)$ of the integral equation (18) with $\sigma = \sigma^*$, which is positive on $(0, 1)$ and vanishes at $s = 1$, cannot be the maximal solution of the integral equation for this specific value of σ. Hence, by the previous paragraph, θ^* must satisfy (23). On the other hand, elementary computation shows that θ^* conforms to (21). Thus, we have a contradiction. The only option is to concede that $\sigma_c = \sigma^*$.

To complete the proof of the theorem we use the information that for every $\sigma > \sigma_c$ the solution $\theta(s)$ of (18) which is positive on (0, 1) and vanishes at $s = 1$ cannot be the maximal solution $\overline{\theta}(s)$ of this equation. Therefore, it satisfies (23). At the other extreme, defining $\Theta(s) := \theta(1 - s)$, we deduce that Θ solves the integral equation

$$\Theta(s) = -\sigma s + \frac{b}{k+1}\{1 - (1 - s)^{k+1}\} + ac\int_0^s \frac{(1 - r)^{k+1}\{1 - (1 - r)^k\}}{\Theta(r)}\,\mathrm{d}r$$

on [0, 1]. Due to the change in sign of the integral term, this equation has a unique solution Θ. A comparison argument similar to those we have previously performed reveals that

$$\Theta(s) \sim \frac{2kac}{\sigma - b + \sqrt{(\sigma - b)^2 + 4kac}}s \quad \text{as} \quad s \downarrow 0.$$

Putting these conclusions in (19) and (20) gives the behaviour of f reported in part (b) of the theorem.

The corollary follows from the theorem through the observation that if f is a solution of (5) satisfying (9) then $\widehat{f}(\xi) := f(-\xi)$ is a solution of problem (3)–(5) with b and σ replaced by $\widehat{b} := -b$ and $\widehat{\sigma} := -\sigma$, respectively.

4. Other cases

When $a > 0$ and $c = 0$, (12) implies that F defined by (13) must be constant. An argument similar to that used previously subsequently gives (11) in this case too. Hence, recalling (3) and (4), we must have $F \equiv \sigma - b/(k+1) = 0$. Thus, $\sigma = b/(k+1)$, and, $a(f^{k+1})' = -bf(1-f^k)$. Observing that the derivative f' exists classically wherever $f > 0$, this yields the following.

Theorem 2. *Suppose that $a > 0$, $c = 0$ and $k > 0$.*

(i) If $b \leqslant 0$ then problem (3)–(5) has no solution.
(ii) If $b > 0$, then, modulo translation, problem (3)–(5) has a unique solution f for $\sigma = b/(k+1)$ and no solution for $\sigma \neq b/(k+1)$. When it exists, f satisfies the conclusions of part (a) of theorem 1.

When $a = 0$ and $c > 0$, equation (5) is nothing more than a first-order ordinary differential equation. Observing that if f is a solution, f' exists classically wherever $\sigma - bf^k \neq 0$, we deduce that any solution satisfying (3) and (4) must be such that $\sigma - bf^k > 0$ where $0 < f < 1$. Furthermore, in such a region we can rewrite the equation as

$$\left(\frac{\sigma}{f} - \frac{(b-\sigma)f^{k-1}}{1-f^k}\right)f' = -c.$$

This gives the following.

Theorem 3. *Suppose that $a = 0$, $c > 0$ and $k > 0$.*

(i) If $b < 0$ then, modulo translation, problem (3)–(5) has a unique solution f for $\sigma \geqslant 0$ and no solution for $\sigma < 0$. When $\sigma = 0$, f satisfies the conclusions of part (a) of theorem 1 with $\frac{\sigma k}{a}$ replaced by $\frac{kc}{|b|}$. Otherwise, it satisfies those of part (b).

(ii) If $b = 0$*, then, modulo translation, problem (3)–(5) has a unique solution* f *for* $\sigma > 0$ *and no solution for* $\sigma \leqslant 0$*. When it exists, modulo translation,*

$$f(\xi) = \left[1 + \exp\left(\frac{ck}{\sigma}\xi\right)\right]^{-1/k}.$$

(iii) If $b > 0$ *then, modulo translation, problem (3)–(5) has a unique solution* f *for* $\sigma \geqslant b$ *and no solution for* $\sigma < b$*. When* $\sigma = b$*, the support of* $1 - f$ *is bounded below, and, modulo translation,*

$$f(\xi) = \begin{cases} 1 & \text{for} \quad \xi \leqslant 0 \\ \exp\left(-\frac{c}{b}\xi\right) & \text{for} \quad \xi > 0. \end{cases}$$

Otherwise, f *satisfies the conclusions of part (b) of theorem 1.*

The existence results contained in parts (i) and (iii) of the above theorem are covered among other results in [20].

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